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Preliminary Study of Periodic Orbits of Interest for Moon Probes. II

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Under the approximation of the restricted three-body problem, two families of periodic orbits that enclose both the earth and the moon in the plane of the earth–moon orbit have been derived and their initial conditions tabulated. Their stability is examined by investigating the variation of the difference of two successive periods with cycles. In the last section, two sequences of initial conditions have been studied in order to find some periodic orbits in the three-dimensional case. Among those investigated, none with the desired characteristic, namely of enclosing both the earth and the moon, has been found. AUTHOR

I. TWO FAMILIES OF DESIRED ORBITS

IN a previous paper (Huang 1962) in which some of the symbols used here have been defined, we have studied, under the approximation of the restricted three-body problem and by means of numerical processes, some interesting orbits which may provide a useful background for deriving periodic orbits for the moon-probing vehicle in the actual earth–moon–sun system. While it has been pointed out that two families of periodic orbits exist—one of direct motion and the other of retrograde motion—for the case $P/P_0=2/3$, only one orbit in each family has actually been given in the previous paper. In the present section we shall give a series of orbits in each family, which are obtained with the aid of an IBM 7090 computer.

Previously, the method of successive approximation was based on the idea that the third body, in a periodic orbit, should recover its initial position and velocity after a certain period of time. Such an approach, though direct and obvious, is not the most efficient. A new scheme is thereby introduced by which the desired orbits can be derived rapidly.

In order to see this new scheme, let us define the period p of a *nearly* periodic orbit by the time interval between two successive crossings over the x axis by the third body at points near to each other. It is apparent that the period thus defined for a nearly periodic orbit changes from cycle to cycle. Thus, we may denote $p_1, p_2, p_3, \ldots, p_n, \ldots$ as the periods of different cycles. All of them may be obtained by interpolation from the results of integration of the equations of motion. Needless to say, we should have $p_1=p_2=p_3=\ldots=p_n=\ldots$ for the true periodic orbits. Thus, the following quantity:

$$\Delta_{n+1,n}=p_{n+1}-p_n, \tag{1}$$

measures the deviation from the periodic orbit.

Following the previous paper, we specify the initial conditions as

$$x=x_0,\quad y=0,\quad \dot{x}=0,\quad \dot{y}=\dot{y}_0, \tag{2}$$

and integrate the equations of motion many times for a fixed value of x_0 but with a series of values for $\dot{y}_0$ which are different from one another only slightly. For each $\dot{y}_0$ we obtain a value $\Delta_{2,1}$ after integrating the equations up to $t>p_1+p_2$. The desired orbit is obtained by finding the value of $\dot{y}_0$ which makes $\Delta_{2,1}$ vanish, a process being performed by interpolation. Since $\Delta_{2,1}$ is very sensitive to the change in $\dot{y}_0$, the process of successive approximation operated in this way converges rapidly. Thus, we have determined six periodic orbits for each family. They are so chosen that when the third body is on the far side of the moon, the closest distance between them lies between 0.08 and 0.20. The results of computation are given in Table I for the direct orbits and in Table II for the retrograde orbits. The integration was performed by the fourth-order Runge–Kutta method with double precision on the IBM 7090, with $\Delta t=0.0078125$. Therefore, the constant of motion maintains its

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TABLE I. Initial conditions for periodic orbits in direct motion.

x_0	$\dot{y}_0$	p_1	$\Delta_{2,1}$
−0.39215	−1.6102480	11.643597	−0.000457
	−1.6102481	11.643566	−0.000380
−0.37215	−1.7014694	11.780075	−0.000091
	−1.7014695	11.780050	+0.000453
−0.35215	−1.7982280	11.900590	−0.000175
	−1.7982281	11.900571	+0.000161
−0.33215	−1.9013902	12.004325	−0.000025
	−1.9013903	12.004311	+0.000178
−0.31215	−2.0120181	12.092292	−0.000105
	−2.0120182	12.092282	−0.000014
−0.29215	−2.1314275	12.166330	−0.000034
	−2.1314276	12.166323	+0.000036

constancy in every case for at least six significant figures for a time interval of 4π. Lagrange four-point inverse interpolation was then used to compute the value of p_n from the integrated values of $y(t)$.

For each value of x_0 in both tables, two values of $\dot{y}_0$ are given—one with $\Delta_{2,1}$ being positive and the other with $\Delta_{2,1}$ negative. Therefore, the correct value of $\dot{y}_0$ for each x_0 that corresponds to the periodic orbit must lie somewhere between these two tabulated values. Here we can clearly see how sensitive $\Delta_{2,1}$ is with respect to the change of $\dot{y}_0$.

Needless to say, the present procedure does not provide an analytic proof of the existence of the computed families of orbits. However, it does give an intuitive assurance that such families may perhaps exist in a rigorous sense.

II. STABILITY CONSIDERATIONS

The stability of a periodic orbit in the restricted three-body problem may be learned from what Poincaré has called characteristic exponents in the solution of the Hill equation which has been recently studied by Message (1959). It is not easy to study it. Here we shall present a more intuitive and much easier way for demonstrating the stability or the instability of a periodic orbit by investigating the change in period, i.e., $\Delta_{n+1,n}$, of the $(n+1)$th cycle from the nth cycle.

If a periodic orbit is stable, any orbit in its neighborhood deviates from it always by a small amount as time increases. Consequently, $\Delta_{n+1,n}$ must oscillate without increasing its amplitude with respect to the increase of n (which is equivalent to the increase in t). On the other hand, if the amplitude of $\Delta_{n+1,n}$ increases with n, the periodic orbit cannot be stable. Hence the problem of stability reduces to one of investigating the behavior of $\Delta_{n+1,n}$ with respect to n, which can be, of course, obtained in the course of numerical integration of the equations of motion.

In the previous paper we have demonstrated by graphs that the direct orbit is unstable while the retrograde orbit is stable. Here we shall illustrate quantitatively the instability of the direct orbit and the stability of the retrograde orbit by tabulating $\Delta_{n+1,n}$. Table III lists the successive periods p_n for two values of $\dot{y}_0$ in the next to the last case in Table I. Here we see $\Delta_{n+1,n}$ fluctuates from positive to negative and vice versa with increasing amplitudes until its magnitude is so large that the orbit can no longer be regarded as nearly periodic. Indeed, the increase in amplitude of $\Delta_{n+1,n}$ with respect to n can be fairly represented by an exponential function as the ratio of the two consecutive Δ's, i.e., $|\Delta_{n+1,n}/\Delta_{n,n-1}|$ as given in the third and fifth column is approximately equal and is greater than one. This clearly shows the characteristic of instability.

TABLE III. Variation in period for the direct orbit ($x_0=-0.31215$).

	$\dot{y}_0=-2.0120181$		$\dot{y}_0=-2.0120182$	
n	$\Delta_{n+1,n}$	$\lvert\Delta_{n+1,n}/\Delta_{n,n-1}\rvert$	$\Delta_{n+1,n}$	$\lvert\Delta_{n+1,n}/\Delta_{n,n-1}\rvert$
1	−0.000105		+0.000014	
2	+0.000909	8.7	−0.000131	9.4
3	−0.007777	8.6	+0.001124	8.6
4	+0.066079	8.5	−0.009618	8.6
5	−0.644121	9.7	+0.081658	8.5
6			−0.841578	10.3

On the other hand, the periodic orbits in retrograde motion are stable, because $\Delta_{n+1,n}$ oscillates with a nearly constant amplitude. For example, the variation in $\Delta_{n+1,n}$ with n for the case ($x_0=-0.43215$) which corresponds to the fourth case given in Table II, is listed in Table IV for four different values of $\Delta\dot{y}_0$ and where $\Delta\dot{y}_0$ represents the deviation of $\dot{y}_0$ from the correct value of the periodic orbit. For each $\Delta\dot{y}_0$, the oscillation of $\Delta_{n+1,n}$ does not increase in amplitude as n or t increases. Moreover, the amplitude becomes

TABLE II. Initial conditions for periodic orbits in retrograde motion.

x_0	$\dot{y}_0$	p_1	$\Delta_{2,1}$
−0.49215	2.1636769	12.836882	+0.000002
	2.1636770	12.836887	−0.000013
−0.47215	2.1937505	12.791218	+0.000009
	2.1937506	12.791223	−0.000005
−0.45215	2.2272677	12.755097	+0.000004
	2.2272678	12.755102	−0.000011
−0.43215	2.2645059	12.726637	+0.000015
	2.2645060	12.726642	0.000000
	2.2645061	12.726648	−0.000017
−0.41215	2.3058168	12.704262	+0.000010
	2.3058169	12.704268	−0.000007
−0.39215	2.3516410	12.686682	+0.000009
	2.3516411	12.686689	−0.000011

TABLE IV. Variation in period ($\Delta_{n+1,n}$) for the retrograde orbit ($x_0=-0.43215$).

n \ $\dot{y}_0$ / p_1	2.2645060 / 12.726642	2.2645065 / 12.726670	2.2645070 / 12.726698	2.2645075 / 12.726726
1	0.000000	−0.000079	−0.000158	−0.000237
2	+0.000002	+0.000082	+0.000164	+0.000246
3	−0.000003	−0.000006	−0.000012	−0.000018
4	−0.000001	−0.000077	−0.000152	−0.000227
5	+0.000004	+0.000087	+0.000169	+0.000252
6	−0.000003	−0.000013	−0.000021	−0.000034
7	+0.000002	−0.000073	−0.000149	−0.000217
8	0.000000	+0.000089	+0.000177	+0.000259
9	−0.000003	−0.000022	−0.000039	−0.000054
10	−0.000001	−0.000066	−0.000136	−0.000202
11	+0.000006	+0.000089	+0.000178	+0.000263
12	−0.000005	−0.000027	−0.000046	−0.000072
13	+0.000001	−0.000059	−0.000130	−0.000186

smaller and smaller as the deviation $\Delta\dot{y}_0$ decreases. This is, of course, the characteristic of a stable orbit.

III. THREE-DIMENSIONAL CASE

We have tried to find some periodic orbits on which the third body encounters the moon at regular intervals. The introduction of the third dimension into the problem broadens the choice of the initial conditions and consequently complicates the processes of finding the desired orbits. We have, therefore, to limit our choice of initial conditions to the following combinations:

$$\begin{aligned} &x=x_0, \quad y=0, \quad z=0, \\ &\dot{x}=0, \quad \dot{y}=\dot{y}_0, \quad \dot{z}=\dot{z}_0. \end{aligned} \tag{3}$$

Even with this limitation we still find that there are too many initial conditions to be studied. We have chosen among them two sequences given by

$$\begin{aligned} &(1) \quad x_0=-0.39215, \\ &(2) \quad x_0=-\dot{y}_0. \end{aligned} \tag{4}$$

In the first case the sequence is obtained by varying $\dot{y}_0$ while in the second case by varying x_0. The second sequence corresponds to the launching of the third body with a velocity perpendicular to the plane of orbit of the moon.

Generalizing the procedure we have proposed in the first section, we now define a period of the nth cycle with respect to the xy plane, denoted by $p_n(z=0)$, as the time interval between the $(n+1)$th and the nth passages of the third body through the xy plane at x near to the initial value x_0. Similarly, we define the period of the nth cycle with respect to the xz plane, denoted by $p_n(y=0)$ as the interval between the $(n+1)$th and the nth passages of the third body through the xz plane at x near to x_0.

Let us now consider the first sequence by assigning numerical values to $\dot{y}_0$ and $\dot{z}_0$, and integrate the equations of motion. It is evident that for an arbitrary pair of values for $(\dot{y}_0,\dot{z}_0)$, $p_1(z=0)\neq p_1(y=0)$. Therefore, we fix $\dot{y}_0$ and vary $\dot{z}_0$ until

$$p_1(z=0)=p_1(y=0) \tag{5}$$

is satisfied. In this way one value of $\dot{z}_0$ is obtained for each fixed value of $\dot{y}_0$. Similarly, we have found one value z_0 for each given value of x_0 in the second sequence that leads to Eq. (5). In Tables V and VI we have given the results of our computation. All pairs of initial conditions given here yield to orbits that satisfy Eq. (5). Since this is only a probing investigation of periodic orbits for moon probes in the three-dimensional case, computation was performed by the machine only with single precision. Consequently fewer significant figures are given in these two tables than in other tables in this paper.

TABLE V. First sequence of initial conditions which lead to $p_1(z=0)=p_1(y=0)$ ($x_0=-0.39215$).

$\dot{y}$	$\dot{z}$	$x(y=z=0)$	p_1
0.075	1.88403	−0.3916	12.5757
0.105	1.75045	−0.3921	12.5655
0.135	1.75491	−0.3921	12.5661
0.165	1.82376	−0.3885	12.5583
0.195	1.89940	−0.3917	12.5772
0.225	1.97678	−0.4169	12.5408
0.255	1.97886	−0.4167	12.5419
0.285	1.98049	−0.4165	12.5428
0.315	1.98166	−0.4164	12.5438
0.345	1.98237	−0.4162	12.5448
0.375	1.98263	−0.4160	12.5457
0.405	1.98244	−0.4158	12.5467
0.435	1.98179	−0.4156	12.5476
0.465	1.90610	−0.3919	12.5810
0.495	1.97913	−0.4151	12.5494
0.525	1.83067	−0.3892	12.5653
0.555	1.69759	−0.3920	12.5688
0.585	1.63092	−0.3922	12.5694
0.615	1.75632	−0.3921	12.5756
0.645	1.75205	−0.3921	12.5762
0.675	1.88475	−0.3921	12.5846

TABLE VI. Second sequence of initial conditions which lead to $p_1(z=0)=p_1(y=0)$.

x_0	$\dot{z}_0$	$x(y=z=0)$	p_1
−0.39215	1.98258	−0.4159	12.5463
−0.37215	2.05440	−0.3927	12.5479
−0.35215	2.13174	−0.3695	12.5495
−0.33215	2.21547	−0.3467	12.5510
−0.31215	2.24207	−0.3120	12.5850
−0.29215	2.34458	−0.2921	12.5863

The condition given by Eq. (5) is not sufficient to warrant a periodic orbit. In fact, most are not. In order to be periodic they must furthermore satisfy either the conditions $p_1(z=0)=p_1(y=0)=4\pi$ and $x(t=4\pi)=x_0$ or the conditions

$$p_1(z=0)=p_1(y=0)=p_2(z=0)=\cdots$$
$$=p_n(z=0)=p_n(y=0)=\cdots.$$

We have examined only cases which are to satisfy the first set of conditions, i.e., the period is equal to exactly two complete revolutions of the moon around the earth. These conditions make the actual search for periodic orbits in the three-dimensional case extremely tedious. On the other hand, if a periodic orbit should be derived in the framework of the restricted three-body problem, perhaps we could expect that a similar orbit may be derived in the actual system of the earth and the moon by modifying the initial conditions.

When we examine the variations of p_1 we find that some initial conditions must exist which lead to

$$p_1(z=0)=p_1(y=0)=4\pi.$$

For example, the desired initial conditions perhaps lie near $\dot{y}_0=0.135$ in the first sequence. However, the orbits which are obtained by these initial conditions do not pass around the far side of the moon, although it appears that they are stable in the sense defined in the previous paper.

For the second sequence we obtain two values of $\dot{z}_0$ for each x_0 that lead to $p_1(y=0)=p_1(z=0)$ but neither of them is equal to 4π for the range of x (from -0.29215 to -0.39215) considered.

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